

Electromagnetic Forces

Sidney Bertram, Ph.D., Fellow IEEE

ABSTRACT

It is assumed that electrostatic forces are developed in the charge fields, where the energy is, and that the fields are made up of elements that move randomly at the speed of light. The mutual energy developed in the overlapping fields of two charges changes when one charge is moving. When a charge accelerates induced voltages are developed since the energy is changing. When two interacting charges are in motion the cross-product terms in the energy equations lead to the magnetic forces while the square terms are canceled by the effects of the ions left behind in the conductors by the moving charges. The derivations are classical.

INTRODUCTION

The following derivations of the electromagnetic forces evolved from a course in electromagnetic theory taught by the author at The Ohio State Graduate Extension at Wright Field in 1946-7. At that time, he vowed to develop a better way of introducing magnetic forces. He left the academic world soon after for electronic development work, but the challenge remained and he worked on it in his spare time and on into his retirement years. He finally found the derivations described herein that he considers eminently satisfactory.

LOCAL INTERACTIONS AND MAGNETIC FORCES

The derivations are intended to provide a theoretical foundation for the classical electromagnetic theory that evolved from experiments with constant or slowly changing currents in closed circuits. The related fields are therefore essentially stationary so propagation delays are not considered.

The Transverse Motional Mutual Energy

It is assumed that charge fields are made up of exceedingly tiny elements that move with the velocity of light, c , as they interact with the elements of other charges.

Fig. 1 shows charge q moving with velocity v transverse to the line to a stationary charge q' . A small cylindrical volume, Q' , of axial length l in the direction of the electrostatic force on q' is fixed in the field of q' . It is assumed that the elements of the field of q that pass completely through Q' contribute to the effective mutual energy with q' in proportion to the time they spend in it.

With q stationary, those elements that move directly through Q' would be in it for a time $t = l/c$. With q moving transversely with velocity v only the elements associated with it that have a velocity component $-v$ with respect to q would pass completely through it; they would have an effective velocity through Q' of $c\sqrt{1 - v^2/c^2}$, where $v = v/c$, so the time they spend in it is $t = l/c\sqrt{1 - v^2/c^2}$ so

$$t = \frac{l}{c\sqrt{1 - v^2/c^2}} \quad (1)$$

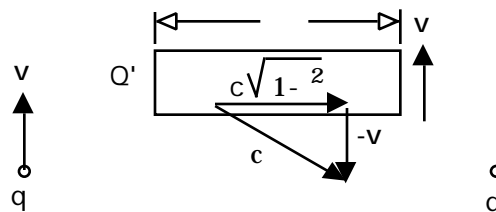


Fig. 1 Charge q moving with velocity v transverse to the line to charge q' .

This increase in interaction time causes a corresponding increase in the mutual energy developed in Q' . Since Q' could have been anywhere in the field, the integrated mutual energy effective on q' is

$$M = \frac{qq'}{4\epsilon_0 r \sqrt{1 - \beta^2}} \quad (2)$$

For the very slow charge velocities characteristic of the electrical circuits used in establishing the classical theory, (2) is approximated by

$$M = \frac{qq'}{4\epsilon_0 r} \left(1 + \frac{v^2}{c^2} \right)$$

The one in the bracket represents the static mutual energy. The motional mutual energy is therefore

$$M = \mu_0 \frac{qq'}{8r} v^2 \quad (3)$$

$\mu_0 = 1/\epsilon_0 c^2$ is called the *permeability* of space.

Establishing the magnetic forces between two charges when one is moving along the radial to the other and the second is moving transversely with respect to it, required to establish the cross-product current-force relationship, requires the motional mutual energy for longitudinal motions. It is derived later.

Induced forces

Since the mutual energy between two charges is a function of their relative velocity there are related forces as a charge accelerates. We say such forces are electrical since they do not involve motion of the second charge.

Fig. 2 shows charge q with an x -directed constant acceleration a . It is assumed to have come in from a very large x , came to a stop at $x=0$ where it was a distance r along the perpendicular to the motion from a second charge q' , and then moved back to the right. When x was near zero the velocity of q , $v = \sqrt{2ax}$, was essentially transverse to the radial to q' , so (3) applies. Thus, for very small x , the mutual energy with the x -directed acceleration a of q in the field of q' , is

$$M_a = \mu_0 \frac{qq'}{4r} ax \quad (4)$$

Since M_a increases linearly with x there must be a positive x -directed force acting on q and a reaction or *induced* force on q' of

$$f_a = -\mu_0 \frac{qq'}{4r} a \quad (5)$$

With interest centered on the forces on conductors, q can be considered to be a charge element $\rho_l dl$ on a differential conductor of length dl , where ρ_l is the charge per unit length of the conductor. If the charges are moving with velocity v along the conductor, the differential current element is $\rho_l v dl = i dl$. Borrowing the classical notation, the induced force associated with transverse accelerations can be written in the form

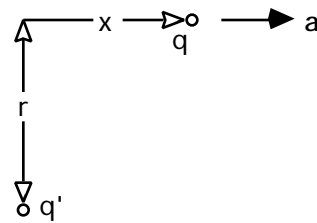


Fig.2 Charge q , accelerating to the right near $x=0$, has velocity and acceleration nearly transverse to line to q' .

$$f' = -q' \frac{A}{t} \quad (6) \quad \text{where} \quad dA = \mu_0 \frac{idl}{4 r} \quad (7)$$

The dA in (7) is the differential magnetic vector potential that integrates to yield the A of (6). Equations (6) and (7) are the differential equivalent of Faraday's law when motion of the conductors is not involved.

Note that radial accelerations do not introduce induced forces since replacing the x in (4) by r would make the mutual energy independent of r .

Magnetic forces

The charges forming the currents in closed circuits are accompanied by the fixed ions they left behind in the conductors. Both sets of charges are involved in the development of magnetic forces.

Two transversely moving charges

Fig. 3 shows charges q and q' moving with parallel velocities v, v' at a time when their separation is transverse to their velocities. Charges $-q$ and $-q'$, shown in parentheses, are ionized molecules left behind by the moving charges; they have zero velocity.

The motional mutual energy for this situation involves the charge pairs: (moving q , moving q' with relative velocity $[v-v']$), (moving q stationary $-q'$ with relative velocity v) and (moving q' , stationary $-q$ with relative velocity v'). The (moving q , stationary $-q$ with relative velocity v), and (moving q' stationary $-q'$ with relative velocity v') do not affect the external forces.

Using (3), the energy associated with the motion that leads to *magnetic* forces is

$$M = \frac{\mu_0}{8 r} qq' (v - v')^2 - q'qv^2 - qq'v'^2 . \quad (8)$$

This reduces to

$$M = -\mu_0 \frac{(qv)(q'v')}{4 r} . \quad (9)$$

While (9) suggests that the mutual energy effective in producing magnetic forces is only that between the moving charges, the ionized molecules left behind provide a significant contribution -- they cancel the energy associated with the squared velocities. Since the mutual energy is negative for charges of the same sign moving in the same direction, (9) leads to attractive forces between the paired charges of Fig. 3 of

$$f = \mu_0 \frac{(qv)(q'v')}{4 r^2} \quad (10)$$

The force f' on q' is therefore upward.

Charges having orthogonal motions

Fig. 4 shows charge q_r moving along the radial towards q with velocity v_r while charge q is moving transverse to the radial with velocity v . Since the velocities are orthogonal there is no mutual energy that leads directly to magnetic forces. However, forces are present as shown next.

If q_r is displaced perpendicularly to the line

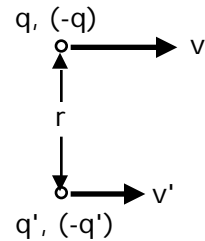


Figure 3 Charges q and q' moving with velocities v, v' transverse to their separation

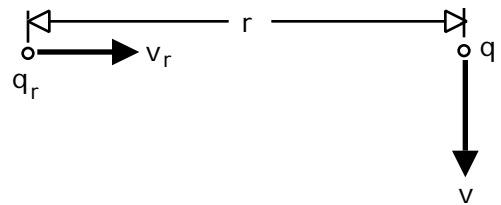


Fig. 4 Charges with orthogonal velocities have no motional mutual energy.

to q (upwards) by a very small amount, exaggerated in Fig. 5, the situation is different; v_r then has a component v_r/r that is essentially parallel to v but with the opposite sense while v has a component v/r that is essentially parallel to v_r . Using (9), the mutual energy of the v_r/r term becomes $M = \frac{\mu_0}{4 r^2} (q_r v_r)(qv)$ which increases with v_r , suggesting that there is a downward force on q_r – backwards from observations. However, as shown next, the contribution of v/r subtracts from this and results in the observed force.

Determination of the contribution of v/r to the motional mutual energy requires a determination of the time the motion associated with q in Fig. 1 remains in the cylinder Q if q is moving along the radial to q' . In this case half the effective elements of q are moving in the direction of v so they have a relative velocity in the cylinder of $(c+v)$, while the other half are moving in the opposite direction so their velocity relative to the cylinder is $(c-v)$. Assuming that their contributions are equal under static conditions, the mutual energy for motion along the radial is

$$M_r = \frac{qq'}{8 r} \left(\frac{c}{c+v_r} + \frac{c}{c-v_r} \right) = \mu_0 \frac{qq}{4 r} \frac{1}{1 - \frac{v_r^2}{c^2}}$$

Since, for low velocities, $\frac{1}{1 - \frac{v_r^2}{c^2}} \approx 1 + \frac{v_r^2}{c^2}$, this leads to a motional mutual energy for radial motion of

$$M_r = \mu_0 \frac{qq}{4 r} v_r^2 \quad (11)$$

This is twice the value for transversely moving charges having the same velocity.

The analysis leading to (9) has a direct counterpart for the radial case with only a change of q' to q_r and v' to v_r . From Fig. 7 it can be seen that for the positive offset of q_r shown, the two radial components are in the same direction, so the mutual energy is negative. Since it is twice the positive contribution of the transverse component, their combined mutual energy is

$$M_r = - \mu_0 \frac{(qv)(q_r v_r)}{4 r} \quad (12)$$

This becomes more negative linearly with v_r , so there is an upward force on q_r of

$$f_r = \mu_0 \frac{(qv)(q_r v_r)}{4 r^2} \quad (13)$$

This is the same as the force on the charge q' of Fig. 5. In terms of current elements the force is

$$f_r = \mu_0 \frac{(idl)(i_r dl_r)}{4 r^2} \quad (14)$$

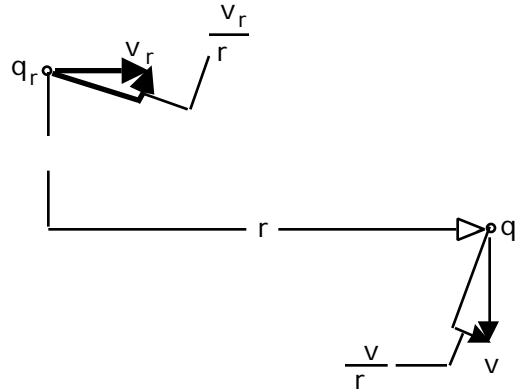


Fig. 5 . Offset orthogonal current elements have mutual energy leading to magnetic forces.

General Charge Motions -- The cross product

Fig. 6 shows current element idl tangent at a general location on a circle centered on current element $i'dl'$. Resolution of $i'dl'$ into components parallel to and perpendicular to the radial to idl shows that there is a normal force on $i'dl'$ equal to the value in (10), so it is independent of where idl is on the circle. It is useful to use the direction of the circle, as described by the cross product of the direction of idl and the unit vector from idl to $i'dl'$ in describing the force. Using the classical notation, the differential expression for the normal force on $i'dl'$ can be written as

$$df = i'dl' \times B \quad (15)$$

where B , the "magnetic flux density" at the location of $i'dl'$, points into the page. The differential magnetic flux density set up by idl is

$$dB = \frac{\mu_0}{4} \frac{i}{r^2} (d\mathbf{l} \times \mathbf{1}_r) \quad (16)$$

Here $\mathbf{1}_r$ is a unit vector from idl to $i'dl'$.

The General Formula for Induced Forces

Changing currents produce induced forces as described by (6) and (7). Also, the factor $i'dl'$ in (15) can be written as $q'v'$ so a charge q moving in a magnetic field with velocity v experiences a magnetic force $qv \times B$. Combining this with the force given by (6) yields

$$\mathbf{f} = q \mathbf{v} \times B - \frac{A}{t} \quad (17)$$

If the v in (17) is associated with the motion of a conductor then $\mathbf{v} \times B$ is equivalent to an electric field that acts on the free charges in the conductor. Considering the bracket in (17) to be the effective E , its line integral around any conducting loop yields Faraday's law that the induced voltage in a loop equals the rate at which the total magnetic flux in a loop is changing.

CONCLUSION

In the opening paragraph of Einstein's 1905 paper¹ he wrote that he undertook his study because of a need to explain why the same voltages are induced in conductors when the conductors move across magnetic fields as when magnetic fields move across conductors. Since the physicists of that era weren't aware that the current flow in conductors consists of free electrons whose motion creates magnetic fields a classical solution of his problem wasn't possible. He solved it by the invention of special relativity. The assumption that the interactions between particles take place in their fields provides the basis for classical derivations of the electromagnetic forces.

While longitudinal currents play no part in current electromagnetic theory, the derivation of the forces between orthogonal current elements requires consideration of both transverse and longitudinal forces. Accelerating longitudinally, charges don't generate induced forces so longitudinal motions do not appear to have other effects in closed circuits.

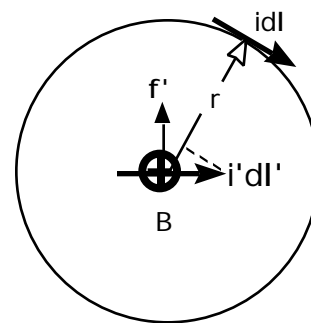


Fig. 6 Force \mathbf{f}' on $i'dl'$ is independent of where idl is on the circle centered on $i'dl'$

¹ A. Einstein, "On the electrodynamics of moving bodies" **The Principle of Relativity**, Dover Publications, Inc. 1952