ROTATION FUNCTIONS AND TRIGONOMETRY

Sidney Bertram, Ph.D., Fellow IEEE

INTRODUCTION

Many practical problems involve vibration. Violin strings vibrate when a bow moves across them, buildings shake during earthquakes and bridges have fallen because high winds made them vibrate until they broke. Also, the alternating current electricity that enters our homes is, in effect, vibrating. The solution of related problems are made much easier by *imagining* that there is a second motion that makes the vibrations look like rotations. The purpose of this note is to introduce the rotation functions that are used in the analysis of such problems and to show that their study offers a useful approach to the study of trigonometry.

ROTATION FUNCTIONS

Fig. 1 shows a disc that is rotating at an angular rate of N revolutions per second. The X,Y-axes were marked on the disc when it was stationary and are now rotating with it, so p, a fixed point on the wheel, has constant x,y (rectangular) coordinates and r,A (polar) coordinates with respect to the axes. (In the following, angles on the disc are initially expressed in degrees with 360° for a complete circle.) The line from p to the origin at O is called a "phasor".

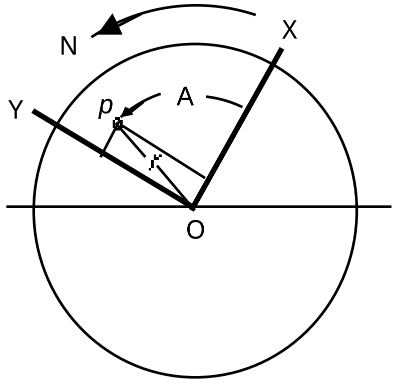

Fig. 1 Rotating disc. Everything rotates together

Fig. 2 shows the development of the "cosine" and "sine" functions. As the point p on the circle of unit radius rotates through 360° on the disc its x-coordinate traces out the curve $x = \cos A$ while its y-coordinate traces out the curve $y = \sin A$.

The rectangular and polar coordinates are related by the cosine and sine functions as described by the equations $x = r\cos A$ and $y = r\sin A$. It can be seen from the figures that $\cos 0° = 1$, $\sin 0° = 0$, $\cos 90° = 0$, and $\sin 90° = 1$. Also, if A is negative the sign of the y component is changed but the sign of the x component is not, so one writes $\sin(-A) = -\sin A$ and $\cos(-A) = \cos A$.

It is useful to represent phasors in their polar and rectangular forms by

$p = r\,\underline{|A}$ (1) and $p = r\cos A + \mathbf{i}\,r\sin A$ (2)

where the **i** in (2) is to be regarded simply as a marker that shows that $r\sin A$ leads $r\cos A$ by 90°. A more useful interpretation of **i** is given later.

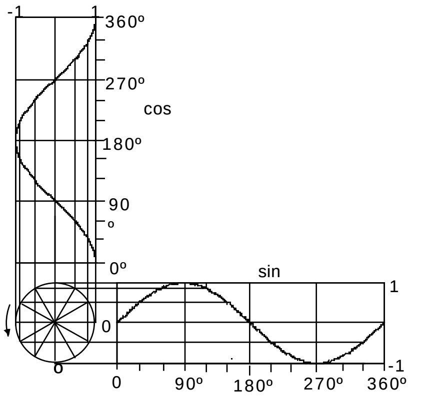

Fig. 2 Development of sinA and cosA functions.

It can be observed from Fig. 2 that the sine curve crosses $y=1$ at about 60°. As angular applications became more technical it became obvious that an angular measure that provides unit slope at the origin would be more useful. For such applications the "radian" is used. The total angle around a circle is 2π radians, so the length of an arc is equal to its radius multiplied by its angle in radians. Radian measure is assumed in the following for such applications; when radians are used Greek letters are used to represent the angles.

PHASOR ALGEBRA

Addition

Fig. 3 shows how phasors are added. Phasor P_b has been copied to the head of phasor P_a to produce their sum $(P = P_a + P_b)$. It is easily seen that this is equivalent to adding their x and y components separately. That is

$$P_a + P_b = (x_a + x_b) + \mathbf{i}(y_a + y_b) \qquad (3)$$

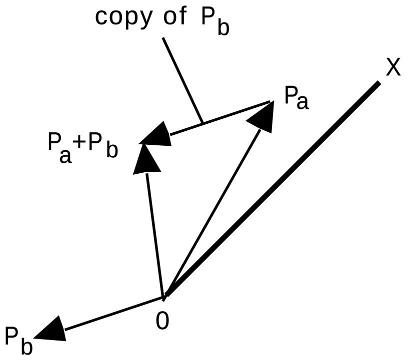

Fig. 3 Addition of phasors.

Subtraction

Reversing the direction of the subtrahend, to create its negative, and adding them, forms the difference of two phasors.

Multiplication

With two numbers available for each phasor, the conventional multiplication algorithm offers no help in determining a useful multiplication algorithm. A thought experiment suggests a useful form for it.

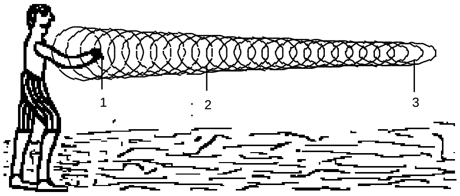

Fig. 4 Operator creates spiral wave on long rope

Fig. 4 depicts an operator who has pulled a rope attached to a distant fixed point taut and is now moving his end of the rope with a constant angular velocity around a circle in a counterclockwise motion, thereby sending a spiral wave down the rope. The horizontal motion of the operator's hand, can be described by $x = R\cos\omega t$ and the vertical motion by $y = R\sin\omega t$ where ω,in radians/second is equal to $2\pi N$, N being the number of circles per second. The combined motion at position 0 can therefore be written as $M = R\angle\omega t$.

At station 1 the motion follows the excitation at station 0 but it is smaller because of losses due to air resistance and bending stresses in the rope. In addition the angular position of the wave lags that at station 0 because of the time it takes the wave to move from station 0 to station 1. Thus one can write $M_1 = MF_{01}\angle{-\alpha_{01}}$ where F_{01} is the fractional loss in amplitude (from zero to one) and α_{01} is the angular delay corresponding to the time delay. As the wave moves from station 1 to station 2 there is an additional fractional loss and another delay. The wave at station 2 can therefore be expressed as $M_2 = MF_{01}F_{12}\angle\alpha_{01} + \alpha_{12}$ where the fractional losses have multiplied and the delay angles have added.

There is therefore a need for an algorithm that multiplies the magnitudes but adds the angles so one defines the multiplication operation for phasors as

$$(R_a\angle\alpha_a)(R_b\angle\alpha_b) = R_aR_b\angle\alpha_a + \alpha_b \qquad (4)$$

Thus the angles behave like exponentials. It is easily seen that both phasor addition and multiplication are commutative.

PHASORS AS ALGEBRAIC EXPRESSIONS

The great utility of **i** lies in the fact that it behaves like an algebraic quantity making it useful in quite complicated situations. Suppose **i** is applied to $\mathbf{i}A$, that is to a length A parallel to the +Y axis, making it $\mathbf{i}(\mathbf{i}A)$. With two 90° rotations from the x-axis $\mathbf{i}(\mathbf{i}A)=\mathbf{i}^2A$ lies in the negative x direction. Thus, one writes $\mathbf{i}(\mathbf{i}A) = \mathbf{i}^2 A = -A$ Assuming that **i** behaves like an algebraic quantity, one is led to $\mathbf{i}^2 = -1$, so **i** can be tentatively equated to the imaginary $\sqrt{-1}$. It must be verified that this is useful in more complicated situations.

The multiplication of unit phasors in rectangular coordinates provides a meaningful test. Using (2) and (4),

$$\left(\cos A + \mathbf{i}\sin A\right)\left(\cos B + \mathbf{i}\sin B\right) = \cos(A+B) + \mathbf{i}\sin(A+B) \tag{5}$$

Performing the indicated multiplications, using $\mathbf{i}^2 = -1$, and realizing that the real (x) and imaginary (y) components must be separately equal, leads to the two equations

$$\cos A \cos B - \sin A \sin B = \cos(A+B) \tag{6}$$

and

$$\cos A \sin B + \sin A \cos B = \sin(A+B) \tag{7}$$

These are easily shown to be correct by using the geometric construction used in classical trigonometry, so the utility of using the imaginary in this way is established.

The Imaginary in Phasor Operations

It is useful in many calculations to express the quantities in rectangular coordinates. The use of phasors simplifies the work. Thus, considering two phasors represented by $(a+\mathbf{i}b)$ and $(c+\mathbf{i}d)$ their sum and product are

$$(a+\mathbf{i}b)+(c+\mathbf{i}d) = (a+c)+\mathbf{i}(b+d) \tag{8}$$

and

$$(a+\mathbf{i}b)(c+\mathbf{i}d) = (ac-bd)+\mathbf{i}(bc+ad) \tag{9}$$

Complex fractions are *rationalized* by multiplying the numerator and denominator by the *conjugate* of the denominator as follows

$$\frac{1}{a+\mathbf{i}b} = \left(\frac{1}{a+\mathbf{i}b}\right)\left(\frac{a-\mathbf{i}b}{a-\mathbf{i}b}\right) = \frac{a-\mathbf{i}b}{a^2+b^2} \tag{10}$$

This has real and imaginary terms proportional to those of the denominator but the angle is the negative of its angle.

THE TRIGONOMETRIC IDENTITIES

Unit phasors provide the basis for an easy determination of the trigonometric identities. For example, setting $B = -A$ in (6) yields

$$\cos^2 A + \sin^2 A = 1 \tag{11}$$

This is the famed Theorem of Pythagoras. Using it and an obvious extension of (5) leads to the multiple angle identities. For example

$$\cos 3A + \mathrm{i}\sin 3A = \left(\cos A + \mathrm{i}\sin A\right)^3 = \cos^3 A + 3\mathrm{i}\cos^2 A \sin A - 3\cos A \sin^2 A - \sin^3 A$$

This reduces to the two equations

$$\cos 3A = 4\cos^3 A - 3\cos A \quad \text{and} \quad \sin 3A = 3\sin A - 4\sin^3 A$$

Digression on Complex Numbers

The square roots (and even fractional roots) of complex numbers are easily obtained using phasors. The rule for multiplying phasors expressed in polar form leads quickly to the square root algorithm. Thus

$$z = \left(\sqrt{R}\ \underline{|A/2}\right)\left(\sqrt{R}\ \underline{|A/2}\right) = R\underline{|A} \quad \text{so} \quad \sqrt{R\underline{|A}} = \left(\sqrt{R}\ \underline{|A/2}\right)$$

Suppose that $z = 4\mathbf{i} = 4\ \underline{|90^\circ}$; then $\sqrt{z} = 2\ \underline{|45^\circ} = \sqrt{2}\left(1+\mathbf{i}\right)$. Note that 4i can also be represented by $4\ \underline{|450^\circ}$ so there is a second root of $\sqrt{4\mathbf{i}} = 2\ \underline{|225^\circ} = -\sqrt{2}\left(1+\mathbf{i}\right)$. To determine the n^{th} root of any number convert it to its polar form, determine the n^{th} root of its magnitude and divide the angle by n. To determine the additional roots add (n-1) multiples of 360° to the original angle and divide by n.

ADVANCED APPLICATIONS

While phasors are useful in deriving the trigonometric identities their real application is to more advanced problems. Two applications are derived in this section. For the purpose the angle is expressed in the form of an exponential function of the time, that is, as $e^{\mathbf{i}\omega t}$ where ω is expressed in radians per second; the multiplication algorithm suggests the exponential form; the value of e must be determined. Write $e^{\mathbf{i}\omega t} = \cos\omega t + \mathbf{i}\sin\omega t$. Differentiating this yields

$$\frac{de^{\mathbf{i}\omega t}}{dt} = \omega\left(-\sin\omega t + \mathbf{i}\cos\omega t\right) = \mathbf{i}\,\omega e^{\mathbf{i}\omega t}$$

The derivative of $e^{\mathbf{i}\omega t}$ is therefore equal to $\mathbf{i}\omega e^{\mathbf{i}\omega t}$. This is the value given by the calculus if $e=2.71828\ldots$ The identity of e can also be determined by expanding $e^{\mathbf{i}\omega t}$ in a power series; this yields $\cos\omega t$ as the real part and $\sin\omega t$ as the imaginary part. Note that $\cos\omega t = \dfrac{e^{\mathbf{i}\omega t} + e^{-\mathbf{i}\omega t}}{2}$ and $\sin\omega t = \dfrac{e^{\mathbf{i}\omega t} - e^{-\mathbf{i}\omega t}}{2\mathbf{i}}$.

Electrical Circuits

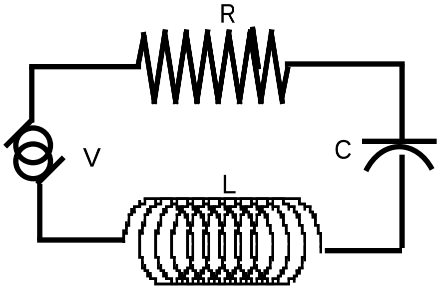

Fig. 4 Electrical circuit with series resistance, capacitance and inductance

Fig. 4 shows a series electrical circuit supplied by an ac voltage v (instantaneous values are denoted by lower case letters and the magnitudes of the sinusoids by capital letters). It is desired to determine the current i through the circuit.

For the purpose the voltage is assumed to be $v = V\cos\omega t$ = real $Ve^{j\omega t}$ where the imaginary is written as $\mathbf{j}$ because i is used for the current in electrical circuits. The circuits are assumed to be linear, so the current is of the same form as the voltage; real voltages give rise to real currents while imaginary voltages give rise to imaginary currents. The resistance R (ohms) has a voltage iR across it. The capacitance C has a voltage across it equal to its total charge (the integral of the current) divided by C (in farads), so letting the voltage be v_C = real part of $V_C e^{j\omega t}$, the current must satisfy

$$V_c e^{j\omega t} = \int \frac{I e^{j\omega t} dt}{C} = \frac{I e^{j\omega t}}{j\omega C} \tag{12}$$

so we write $I = j\omega C V_C$.

The voltage across an inductance L is equal to its value in henries times the rate of change of the current. Thus

$$V_L e^{j\omega t} = L \frac{dI e^{j\omega t}}{dt} = j\omega L I e^{j\omega t} \tag{13}$$

so we write $I = V_L / j\omega L$.

For the circuit of Fig. 4 the applied voltage must equal the sum of these three voltages. This leads to the following equation for the current:

$$I = \frac{V}{R + j\left(\omega L - \frac{1}{\omega C}\right)} = V \frac{R - j\left(\omega L - \frac{1}{\omega C}\right)}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{14}$$

Multiplying this by $e^{j\omega t}$ leads to the instantaneous current i whose real part is

$$i = \frac{V\left(R \cos\omega t + \left(\omega L - \frac{1}{\omega C}\right)\sin\omega t\right)}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{15}$$

Since the sine function reaches its peak $\pi/2$ after the cosine function this shows that the reactive current is delayed by $\pi/2$ radians from the resistive current. Since we can infer this directly from (14) there is no reason to obtain (15).

This is a tremendous simplification of the problem. More complicated ac circuits are handled like the corresponding dc circuits except for the imaginary used with the inductance and capacitance components.

The Single Phase Induction Motor

3600-RPM single-phase induction motors consist of a stator with two sets of windings that are 90° apart in the frame and a rotor that has induced in it from the stator the currents required to produce the magnetic fields that produce the torque. Each stator winding is divided into two parts that are mechanically 180° apart. One stator winding is supplied directly from the 60 Hz line while the other is supplied from the line in a manner that produces a somewhat out-of-phase current; the out of phase component is responsible for the torque, so the in-phase component is neglected in the following. (The reactive current in some motors is obtained by putting a capacitor in series with the line, in others by making the starting winding have a higher resistance than the running winding; in motors with starting windings a centrifugal switch cuts out the starting winding as the motor picks up speed.)

It will be assumed that the $\cos\theta$ field is fed a $\cos\omega t$ current and the $\sin\theta$ field is fed a $\sin\omega t$ current. This results in a total stator field proportional to

$$M = \cos\theta \cos\omega t + \sin\theta \sin\omega t$$

Using (5) and (12) this can be written as the real part of $e^{i(\theta-\omega t)}$ which indicates that there is a sinusoidal magnetic field in the stator that is rotating at a rate of ω radians per second (the zero of the wave satisfies $\theta = \omega t$).

This rotating field cuts across the conductors of the rotor inducing voltages in it. The resulting current, acting on the stator field, creates a torque that brings the motor near the speed of the stator field (synchronous speed).

If the power is removed from the $\sin\theta$ winding after the rotation starts the motor will continue to pick up speed because the $\cos\theta$ winding will then produce the field $(e^{j(\theta-\omega t)}+e^{-j(\theta-\omega t)})/2$ that has both positive and negative rotating components. The negative rotating component will induce a higher-frequency voltage in the rotor so there will be less current in it and less torque than that in the starting direction. If the starting winding of an induction motor burns out the motor can be started in either direction by giving the rotor a slight spin and it will then pick up speed and run properly in that direction. (In like manner, the Hula hoop could be kept going with just a back and forth motion with the centrifugal force holding it to one's body.)

Synchronous motors start as induction motors and lock in to the rotating field because the rotor develops a fixed magnetic field either because of its structure or because a dc current is supplied to it that creates a steady magnetic field in it.

Changing the resistance and inductance characteristics of the rotor varies the starting torque characteristic of induction motors because it affects the rotor current as a function of the frequency as determined by the difference between its speed and the synchronous speed.

CONCLUSION

The use of the imaginary in electrical calculations was popularized by Steinmetz early in the twentieth century and served to simplify the introduction of ac power systems. However it was not universally accepted until much later. The author learned ac circuits in 1931 without the use of imaginaries. Today the imaginary has found a place in many physical theories. It is a very useful device and definitely not just the *imaginary* of the early mathematicians.

A good explanation of the formation of rotating fields eluded the author in 1939 for a class in Rotating Fields (the class included the effects caused by the fact that the fields are never exactly sinusoids). Teaching a class in elementary circuits in 1985 provided the key to its explanation but the two weren't combined until very recently.